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On the $\hbar^2$ correction terms in quantum integrability

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Abstract. We study the question of quantum integrability for two-dimensional Hamiltonian systems with special attention on the $\hbar^2$ correction terms in the potential. A class of Hamiltonians of type $H = \frac{1}{2}(p_x^2 + p_y^2) + c_4 y^4 + y^2[f_2''(x) + \alpha_6] + f_0(x)$ with a second invariant of type $I = p_x^4 + A(x,y)p_x^2 + B(x,y)p_x p_y + C(x,y)p_y^2 + D(x,y)$ is considered. The general solution for f_2 involves elliptic integrals. For quantum integrability the potential must be modified with $\hbar^2$-dependent terms. We construct a point transformation which, after coupling constant metamorphosis, takes the Hamiltonian to a new quantum integrable Hamiltonian for which no correction terms are necessary. The new system does not in general have a flat space kinetic part.

1. Introduction

The question of whether quantum integrability is a consequence of classical integrability has been long standing and is still unanswered in full generality. It is clear, from existing results, that if such a relation holds it is far from trivial. In a series of works [1–3] we have addressed this fundamental question, focusing on two-dimensional Hamiltonian systems and found the following. In all known cases starting from a classical Hamiltonian and an integral of motion (the Poisson bracket of which is zero) one can obtain two commuting quantum operators, which tend to the Hamiltonian and the first integral of the classical limit. However, it has not been possible to formulate a general proposition on this, and the above conclusion was reached by studying each known case individually. In some cases, e.g. whenever the constant of motion is quadratic in the momenta, a straightforward quantisation suffices to give commuting operators. However, not all invariants are p-quadratic, and for some of them no linear quantisation procedure can lead to commuting operators. It has turned out that, even in these cases, we can obtain quantum integrability by adding suitable quantum corrective terms, which are explicitly $\hbar$ dependent, to both the invariant and the Hamiltonian. Such corrections are expected in the invariant due to ordering ambiguities, but it is surprising that they are also needed for flat space Hamiltonians, for which no ordering ambiguity exists. There seems to be no general way of obtaining these correction terms, although the procedure can be systematised in most cases.

The existence of additive terms, especially in the Hamiltonian, is a puzzling phenomenon and one may wonder on their possible origin (a more physical one, something

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more than just being *ad hoc* corrections which restore quantal integrability). In this paper we will present a new viewpoint to this problem.

First in §2 we will introduce a new integrable (two-dimensional Hamiltonian) system, which contains as limiting cases some of the more interesting (for our study) Hamiltonians, namely some of those which necessitate corrective terms in both Hamiltonian and invariant in order to achieve quantum integrability. The origin of this Hamiltonian can be traced back to [4], although due to its complicated form only some of its special limits have been derived previously. Next we proceed to quantise the Hamiltonian and the constant of motion, and obtain the corrective quantum terms. For the Hamiltonian these extra terms are considerably more complicated than the constant $\propto \hbar^2 x^{-2}$ term that has been sufficient before.

In §3 we consider the problem of whether it is possible to eliminate the corrections in the quantum Hamiltonian through the quantum effect of some transformation. It turns out that this can indeed be realised. Thus one can find a canonical' form of the Hamiltonian (where by canonical we mean here the form which can be quantised without additive quantal terms). However, it is one where the kinetic part is, in general, $\frac{1}{2}[p_x^2 + p_y^2 c(x)]$. The corrective terms are therefore due to the fact that we chose to represent the Hamiltonian in flat space, i.e. with kinetic part $\frac{1}{2}(p_x^2 + p_y^2)$. Thus we can make the conjecture that, if correction terms are needed in the Hamiltonian, there exists a transformation to a 'curved space' system, which is integrable without any corrections.

2. New integrable two-dimensional Hamiltonians

The Hamiltonian we are going to derive and quantise in this study belongs to a class of integrable Hamiltonians

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + V(x,y) \tag{1}$$

with a second invariant having the form

$$I = p_x^4 + A(x,y)p_x^2 + B(x,y)p_x p_y + C(x,y)p_y^2 + D(x,y). \tag{2}$$

The problem of finding Hamiltonians belonging to the above class has been addressed in [4], where it has been shown that a necessary condition for (2) to be a conserved quantity is that

$$\partial^4 V/\partial_x \partial_y^3 = 0 \tag{3}$$

or equivalently

$$V = f_3(y) + y^2 f_2''(x) - y f_1'(x) + f_0(x). \tag{4}$$

The condition $\{H, I\}_{\mathrm{PB}} = 0$ yields immediately

$$\begin{aligned} A &= 4[y^2 f_2''(x) + y f_1'(x) + f_0(x)] \\ B &= -4[2y f_2'(x) + f_1(x)] \\ C &= 8 f_2(x). \end{aligned} \tag{5}$$

This still leaves two equations, from which D can be solved if one extra (non-linear) condition is satisfied, a condition which involves the f_i. The general solution to this problem is not known.

In [5] the case $f_2 = 0$ was examined and a family of solutions (although not the general solution) was obtained. Potentials in this family contain as special cases the well known Henon–Heiles [6] and Holt [4,7] potentials. In [4] an analysis of the case with $f_2 \neq 0$ was attempted. Several specific examples of such potentials were found including the quartic potential of [8,9], and generalisations of Hénon–Heiles and Holt potentials. The main progress was that several additive terms to the potentials compatible with integrability were identified. The same approach, and the study of quantum integrability, were the object of [2], where further additive terms were obtained. From the above-mentioned studies it appears clearly that the problem will be quite difficult in its full generality and one must be content with interesting particular solutions.

Thus in the present study we will limit ourselves to the special case of (4) where $f_1 \equiv 0$, $f_3(y) = v_4 y^4 + \alpha_6 y^2$ (v_4 and α_6 are constants), i.e.

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + v_4 y^4 + y^2[f_2''(x) + \alpha_6] + f_0(x). \tag{6}$$

This Hamiltonian has as special limits a quartic Hamiltonian when $v_4 \neq 0$, and the Hénon–Heiles and Holt type models if $v_4 = 0$, and can be thought as 'interpolating' between them.

The condition for I to be a constant of motion is $\{H, I\}_{\mathrm{PB}} = 0$, where $\{.,.\}_{\mathrm{PB}}$ stands for the usual Poisson bracket. However, anticipating the problem of quantum integrability, we can at this stage ask about commutation of the operators associated with H and I. If we assume the Weyl rule for associating variables p_i, q_i with corresponding operators then we can work with c-numbers [2], after replacing the commutator with the Moyal bracket defined by

$$\{A, B\}_{\mathrm{MB}} = \frac{2}{\hbar} A \sin\left(\frac{\hbar}{2} \sum_i (\overleftarrow{\partial}_{q_i} \overrightarrow{\partial}_{p_i} - \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{q_i})\right) B. \tag{7}$$

Here, given the structure of H and I we can expand in powers of $\hbar$ and retain only the non-vanishing terms. With (2) and (6) this leads to

$$\{H, I\}_{\mathrm{MB}} = \{H, I\}_{\mathrm{PB}} - 3\hbar^2 p_x V_{xxx} = 0. \tag{8}$$

The equations for A, B and C do not change so (5) holds as before, but the remaining equations contain $\hbar$. The final equation for integrating D can be solved and leads to an f_0 of the form

$$f_0 = -\tfrac{5}{8}\hbar^2 (f_2''/f_2')^2 + 2f_2 + (4\alpha_6 f_2^2 - 2\alpha_8 f_2 + \alpha_9)/f_2'^2 \tag{9}$$

and the condition that f_2 must be a solution of

$$f_2''''f_2' + 5f_2'''f_2'' - 24v_4 f_2' = 0. \tag{10}$$

Then we get for D the expression

$$D = -2y^4(f_2'''f_2' - 8v_4 f_2) + 4y^2(4f_2''f_2 - f_2'f_0' + 4f_2\alpha_6) - \hbar^2 f_0'' + 4f_0^2. \tag{11}$$

In this way all x dependency has been expressed through f_2. For notational simplicity we drop the subscript 2 from now on.

The differential equation (10) can be integrated three times to

$$f' = (16v_4 f^3 + \beta_2 f^2 + \beta_1 f + \beta_0)^{1/4} \tag{12}$$

where β_i are the integration constants. The final quadrature yields f implicitly by

$$x = \int^f (16v_4 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0)^{-1/4}\, \mathrm{d}s \tag{13}$$

which, upon integrating, allows the calculation of the potential and the invariant. The completion of the integral (13) is, in fact, possible in full generality. Starting with the transformation

$$f = (aW + b)/(cW + d) \tag{14}$$

one can choose parameters a, b, c and d in such a way as to reduce the integral to the form

$$\int (cW + d)^{-1}(gW^4 + kW^2 + m)^{-1/4}\, \mathrm{d}W \tag{15}$$

which can be expressed in terms of elliptic integrals [10].

The final form of f is not very explicit so it is useful to consider some limiting cases. In the case where only one of the constants v_4, β_i is different from zero one gets

$$f = \text{constant} \times x^{4/(4-m)} \qquad \text{with} \qquad m = 0, 1, 2, 3.$$

The cases $f = x$ or x^2 yield separable systems, and since the cross term $B(x, y)p_x p_y$ does not vanish we get a third constant of motion and the systems are 'superintegrable' [11]. If $f = \frac{9}{4}x^{4/3}$ we get the Holt model [4,7]

$$V = \tfrac{9}{2}x^{4/3} + y^2x^{-2/3} + a(9x^2 + 4y^2) + dx^{-2/3} - \tfrac{5}{72}\hbar^2 x^{-2} \tag{16}$$

and if $f = \frac{1}{2}x^4$ we get [2,4]

$$V = 8y^4 + 6x^2y^2 + x^4 + a(x^2 + 4y^2) + dx^{-2}. \tag{17}$$

Another interesting reduction is the one corresponding to $v_4 = 0$. In this case the polynomial in the fourth root in (13) is simply $\beta_2 s^2 + \beta_1 s + \beta_0$ which can be translated and scaled to s^2+1. Furthermore the transformation $s = \sinh(t)$ allows expression of the integral $x = x(f)$ in terms of incomplete elliptic integrals, but unfortunately inversion to $f = f(x)$ is not possible in closed form. There are several other interesting reductions leading to the integral $\int(\cosh(t))^{1/2}\,\mathrm{d}t$, just as in the previous case, after suitable transformations. Among them we distinguish the case where $(f')^4 = (af + b)^2(cf + d)$, the significance of which will become clear in the next section. Indeed, by shifting and rescaling f, the right-hand side becomes $8f(\frac{1}{2} - f)^2$. Then putting $f = \frac{1}{2}\sin(t)^2$ we obtain $x = \int(\sin(t))^{1/2}\,\mathrm{d}t$.

In conclusion, we have obtained a new integrable two-dimensional Hamiltonian system, which contains as limiting cases other well known Hamiltonians and which is also quantally integrable. H and I are polynomials in momenta and the y variable, while the x dependency enters through the function f, defined by (10) or (12). The required quantum correction to the polynomial is $-\frac{5}{8}\hbar^2[f''(x)/f'(x)]^2$.

3. Can the quantum corrections be explained?

In the process of making the classically integrable Hamiltonian also integrable in quantum mechanics we had to introduce some terms explicitly depending on $\hbar$. The next question we will study is the origin of these quantum corrections and whether one can eliminate them.

In [3] we have introduced the concept of the coupling-constant metamorphosis (CCM). This is a (non-canonical) transformation to a 'new time' in which the energy and the coupling constant of some term of the potential are exchanged. We have shown that such a transformation does preserve integrability. As was shown in [3] this type of transformation, coupled with a point transformation, can induce into the Hamiltonian certain correction terms which could be made identical to the quantum integrability corrections.

Let us therefore consider the point transformation

$$u = g(x) \qquad p_u = p_x/g'(x) \tag{18}$$

in the Hamiltonian $H = \frac{1}{2}p_u^2 + K(p_y, u, y)$. Suppose that there exists in $K(p_y, u, y)$ an additive term $-a[g'(g^{-1}(u))]^{-2}$ with a free constant a, i.e.

$$H = \tfrac{1}{2}p_u^2 + L(p_y, u, y) - a[g'(g^{-1}(u))]^{-2} \tag{19}$$

where L is a independent. After the point transformation (18) we multiply (19) with $g'(x)^2$ in order to obtain a standard-type kinetic term in the x coordinate (E is the conserved value of H):

$$Eg'(x)^2 = \tfrac{1}{2}p_x^2 + g'(x)^2 L(p_y, g(x), y) - a. \tag{20}$$

Through CCM the constant a can be taken as the new energy and the new Hamiltonian is

$$A = \tfrac{1}{2}p_x^2 + g'(x)^2 L(p_y, g(x), y) - Eg'(x)^2. \tag{21}$$

This result holds in classical mechanics. Now Pak and Sökmen have shown [12] that, when these transformations are made in quantum mechanics, there will be extra quantum corrections in the potential given by

$$\Delta V_Q = -\frac{1}{4}\hbar^2 \left[\left(\frac{g''(x)}{g'(x)} \right)' - \frac{1}{2} \left(\frac{g''(x)}{g'(x)} \right)^2 \right]. \tag{22}$$

This sets the stage for answering the question we addressed at the beginning of this section: by choosing a suitable point transformation (and performing a CCM) we may change the system to one where the quantum corrections disappear. It is clear from the outset that the price we will have to pay is to have a Hamiltonian expressed no longer in a flat space because of the factor $g'(x)^2$ in L above. Sometimes the new Hamiltonian has y dependency only through y^2; then a further canonical transformation $y = p_v$, $p_y = -v$ yields a Hamiltonian with a conventional flat space kinetic part [3,13]. However, this last step is not necessary from the point of view of eliminating the quantum corrections.

The quantum corrections in the potential at hand are present just in f_0 through the term (9):

$$\Delta V = -\tfrac{5}{8}\hbar^2(f''/f')^2. \tag{23}$$

To find the necessary transformation, let us equate the quantum corrections (22) and (23) :

$$-\frac{1}{4}\hbar^2\left[\left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2\right] = -\frac{5}{8}\hbar^2\left(\frac{f''}{f'}\right)^2 - \hbar^2\frac{2\alpha_8'f - \alpha_9'}{f'^2} \tag{24}$$

where α_8' and α_9' come from the possible $\hbar^2$-dependent parts of the corresponding free constants. (In the other constants α_6, β_i and v_4, such $\hbar^2$-dependent parts would not help.)

In addition to (24) there is another condition necessary for CCM to work completely: there must be a constant in the potential which can be identified with E in (21). From (6) and (9) we see that the constants must come from α_8 or α_9 and we infer the condition

$$[g'(x)f'(x)]^2 = af(x) + b \tag{25}$$

for some constants a and b. In the following we will obtain the complete solution to equations (24) and (25) when f' is given by (12). However, other solutions to (24) alone do exist and they eliminate the quantum corrections, but if we wish to have an underlying Hamiltonian that is integrable at any energy we must require (25) as well.

Let us rewrite equation (12) as $(f')^4 = P(f)$, where $P(f)$ is the cubic polynomial in the RHS, and introduce the derivative with respect to f and denote it by an overdot. If we assume in general that $f'g' = G(f)$ then equation (24), with f as an independent variable, becomes:

$$\frac{\ddot{G}}{G} - \frac{3}{2}\left(\frac{\dot{G}}{G}\right)^2 + \frac{1}{2}\frac{\dot{G}}{G}\frac{\dot{P}}{P} = \frac{\ddot{P}}{4P} + \frac{8\alpha_8'f - 4\alpha_9'}{P}. \tag{26}$$

In the present case with $G = [2(af + b)]^{1/2}$ we obtain

$$-\tfrac{5}{2}a^2P + a(af + b)\dot{P} = (af + b)^2[\ddot{P} + 16(2\alpha_8'f - \alpha_9')]. \tag{27}$$

This has two solutions (when P is a cubic polynomial in f). (i) If $a = 0$ (hence $b \neq 0$, say $b = 1$) we must take $\alpha_8' = -3v_4$ and $\alpha_9' = \beta_2/8$. This works for any P. (ii) If $a \neq 0$ then we find that P must be of the form $P = (af + b)^2(cf + d)$ and then $\alpha_8' = -\frac{11}{64}a^2c$, $\alpha_9' = \frac{5}{32}a^2d + \frac{3}{16}abc$. (This case is one of those where the integration reduces to the integration of $\int(\sin(t))^{1/2}\,dt$, as was mentioned in §2.) Once G is known the integration for g is straightforward. To construct the underlying $\hbar$-independent Hamiltonian we do not need f or g as functions of x but only f as a function of g.

(i) In the case $G = 1$ it is easy to verify (based the well known properties of Jacobian elliptic functions [10]) that

$$f(x) = A + B\,\mathrm{sn}(Cg(x), k)^2 \tag{28}$$

together with $f'g' = 1$ lead to (12), and to find the relation between v_4, β_i and A, B and C. (Let us at this point also note that the Holt potential (16) is obtained in the singular limit $A = 0$, $B \to \infty$, $C \to 0$, with $BC^2 = 9$, while (17) follows when $A = 0$, $B \to 0$, $Ck \to \infty$, with $B^{-1}C^2k^2 = 32$.)

(ii) In the case $G = [2(af + b)]^{1/2}$, $P = (af + b)^2(cf + d)$, $a \neq 0$ we find instead the relation

$$f(x) = -(1/2ac)\{(ad - bc)\cos[(ac/2)^{1/2} g(x)] + (ad + bc)\}. \tag{29}$$

In all cases above a solution to equation (24) is obtained. In principle, any g that solves (24) leads to a cancellation of the quantal corrections in the Hamiltonian: by introducing the appropriate point transformation we can make the quantum-integrability corrections disappear. To each g there corresponds a Hamiltonian (in general with curved space kinetic part) which is integrable without quantum corrections and which leads to (6) with (9) and (10) when the point transformation g followed by CCM is made.

To avoid unnecessary constants let us consider the normalised case only. Using the (inverse of) transformation (18) with CCM and (28) with $A = 0$, $B = \frac{1}{2}$, $C = 1$ (i.e. $P(f) = 8f(f - \frac{1}{2})(k^2 f - \frac{1}{2})$) we arrive at the final expression for the Hamiltonian and the invariant:

$$\begin{aligned} H' = {} & \tfrac{1}{2}p_u^2 + \tfrac{1}{2}\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)p_y^2 + v_4\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)y^4 \\ & + y^2\{v_4[3\,\mathrm{sn}(u)^4 - 2\,\mathrm{sn}(u)^2] - \mathrm{sn}(u)^2 + \tfrac{1}{2} + \alpha_6\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)\} \\ & + \mathrm{sn}(u)^3\,\mathrm{cn}(u)\,\mathrm{dn}(u) - \mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)E + \alpha_6\,\mathrm{sn}(u)^4 - \alpha_8\,\mathrm{sn}(u)^2. \end{aligned} \tag{30}$$

We remark that no explicit $\hbar^2$ terms enter expression (30). Note the appearance of a term $\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)E$ which comes from the exchange of energy and coupling constant. Following the same steps we can obtain the invariant, although it is perhaps easier to derive it directly; the result is

$$\begin{aligned} I = {} & p_y^4 - 8yp_u p_y + 4p_y^2[v_4 y^4 + y^2\alpha_6 + \mathrm{sn}(u)^2 - E] \\ & + 4v_4 y^8 + 8v_4\alpha_6 y^6 - 4y^4\{2v_4[2\,\mathrm{sn}(u)^2 + E - 1] - \alpha_6^2 - 1\} \\ & - 4y^2\{[4v_4\,\mathrm{sn}(u)^6 - (4v_4 + 2)\,\mathrm{sn}(u)^4 + 2\,\mathrm{sn}(u)^2]/[\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)] \\ & + 2\alpha_6[\mathrm{sn}(u)^2 + E] - 2\alpha_8 + 3\hbar^2 v_4\}. \end{aligned} \tag{31}$$

Note that an $\hbar^2$-dependent term persists in the invariant and to eliminate it one might try other methods [2].

For $k^2 = 1$ we can also use the other transformation (ii) as derived earlier (29). We choose the parameters so that they allow a direct comparison with the normalised case above, i.e. $a = -1$, $b = \frac{1}{2}$, $d = 0$ and $c = 8$. This yields $f = \frac{1}{2}\sin(g)^2$ and then the transformed Hamiltonian is

$$\begin{aligned} H = {} & \tfrac{1}{2}p_u^2 + \tfrac{1}{2}\sin(u)p_y^2 + \tfrac{1}{2}\sin(u)y^4 + y^2[\tfrac{1}{2} - \tfrac{3}{2}\sin(u)^2 + \sin(u)\alpha_6] \\ & + \sin(u)^3 - \alpha_6\sin(u)^2 - E\sin(u) + \alpha_{10}\cos(u)^{-2} \end{aligned} \tag{32}$$

with the corresponding invariant

$$\begin{aligned} I = {} & p_y^4 - 4y^2p_u^2 - 8\cos(u)yp_u p_y + \{2y^4 + 4y^2[\alpha_6 - \sin(u)^2] + 4\sin(u)^2 - 4E\}p_y^2 \\ & + y^8 + 4y^6[\alpha_6 - \sin(u)] + 4y^4\{[\alpha_6 - \sin(u)]^2 + 1 - E\} \\ & + y^2\{8(1 - E)[\alpha_6 - \sin(u)] - 5\hbar^2 - 8\alpha_{10}\cos(u)^{-2}\} + 2\sin(u)\hbar^2. \end{aligned} \tag{33}$$

Note that now $\alpha_8 - \alpha_6$ becomes the new energy; $\alpha_{10} = \alpha_6 - \alpha_8 + \alpha_9$.

Thus for $k = 1$ there are two different $\hbar$-independent Hamiltonians to which the original system can be transformed. There should also be a transformation relating them, which produces only such $\hbar^2$-dependent terms which can be absorbed into the free constants. Since $\mathrm{sn}(g, 1) = \tanh(g)$ we have

$$\tanh(g) = \sin(u) \tag{34}$$

as the transformation producing this connection between (32) and (30) (for $k = 1$).

4. Conclusion

In this paper we have investigated the classical and quantum integrability of a two-dimensional Hamiltonian. The potential considered is an even quartic polynomial in the y variable. We have determined the general form of the potential for the Hamiltonian to possess a second constant of the motion of the form p_x^4 plus terms of lower order in the momenta. We have in particular focused on the quantum integrability (i.e. Moyal bracket rather than Poisson bracket vanishing), and have solved the problem and obtained the corrective $\hbar^2$-dependent terms which ensure quantum integrability.

We have also addressed the interesting question about the meaning of these quantum integrability corrections, especially in the Hamiltonian. We have shown that, at least in the case at hand, it is possible to rewrite the Hamiltonian in a curved space (obtained through the introduction of a suitable point transformation combined with CCM) so as to make the corrective terms vanish. Thus these correction terms are the manifestation of the fact that the Hamiltonian possesses a more 'natural' form in a curved space and the corrections are present because we insist on writing the Hamiltonian in a flat space coordinate. This works in principle for all cases where the quantum correction for a flat space Hamiltonian is of the form $\hbar^2[a(x) + b(y)]$.

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